7-7 Solving Radical Equations and Inequalities

Main Ideas

- Solve equations containing radicals.
- Solve inequalities containing radicals.

New Vocabulary

radical equation
extraneous solution
radical inequality

GET READY for the Lesson

Computer chips are made from the element silicon, which is found in sand. Suppose a company that manufactures computer chips uses the formula $C = 10n^{\frac{2}{3}} + 1500$ to estimate the cost C in dollars of producing n chips. This can be rewritten as a radical equation.

Solve Radical Equations Equations with radicals that have variables in the radicands are called **radical equations**. To solve this type of equation, raise each side of the equation to a power equal to the index of the radical to eliminate the radical.

It is very important that you check your solution. Sometimes you will obtain a number that does not satisfy the original equation. Such a number is called an **extraneous solution.**

EXAMPLE 1 Solve Radical Equations

Solve each equation.

a. $\sqrt{x+1} + 2 = 4$

$\sqrt{x+1} + 2 = 4$ — Original equation

$\sqrt{x+1} = 2$ — Subtract 2 from each side to isolate the radical.

$\left(\sqrt{x+1}\right)^2 = 2^2$ — Square each side to eliminate the radical.

$x + 1 = 4$ — Find the squares.

$x = 3$ — Subtract 1 from each side.

CHECK $\sqrt{x+1} + 2 = 4$ — Original equation

$\sqrt{3+1} + 2 \stackrel{?}{=} 4$ — Replace x with 3.

$4 = 4$ ✓ — Simplify.

The solution checks. The solution is 3.

b. $\sqrt{x-15} = 3 - \sqrt{x}$

$\sqrt{x-15} = 3 - \sqrt{x}$ — Original equation

$\left(\sqrt{x-15}\right)^2 = (3 - \sqrt{x})^2$ — Square each side.

$x - 15 = 9 - 6\sqrt{x} + x$ — Find the squares.

$-24 = -6\sqrt{x}$ — Isolate the radical.

$4 = \sqrt{x}$ — Divide each side by -6.

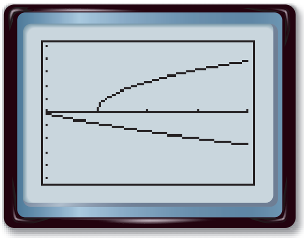
$4^2 = (\sqrt{x})^2$ — Square each side again.

$16 = x$ — Evaluate the squares.

CHECK $\sqrt{x-15} = 3 - \sqrt{x}$

$$\sqrt{16-15} \stackrel{?}{=} 3 - \sqrt{16}$$
$$\sqrt{1} \stackrel{?}{=} 3 - 4$$
$$1 \neq -1$$

The solution does not check, so the equation has no real solution. The graphs of $y = \sqrt{x-15}$ and $y = 3 - \sqrt{x}$ are shown. The graphs do not intersect, which confirms that there is no solution.



[10, 30] scl: 5 by [−5, 5] scl: 1

CHECK Your Progress

1A. $5 = \sqrt{x-2} - 1$ **1B.** $\sqrt{x+15} = 5 + \sqrt{x}$

To undo a square root, you square the expression. To undo an nth root, you must raise the expression to the nth power.

Study Tip

Alternative Method

To solve a radical equation, you can substitute a variable for the radical expression. In Example 2, let $A = 5n - 1$.

$$3A^{\frac{1}{3}} - 2 = 0$$
$$3A^{\frac{1}{3}} = 2$$
$$A^{\frac{1}{3}} = \frac{2}{3}$$
$$A = \frac{8}{27}$$
$$5n - 1 = \frac{8}{27}$$
$$n = \frac{7}{27}$$

EXAMPLE 2 Solve a Cube Root Equation

Solve $3(5n-1)^{\frac{1}{3}} - 2 = 0$.

In order to remove the $\frac{1}{3}$ power, or cube root, you must first isolate it and then raise each side of the equation to the third power.

$3(5n-1)^{\frac{1}{3}} - 2 = 0$	Original equation
$3(5n-1)^{\frac{1}{3}} = 2$	Add 2 to each side.
$(5n-1)^{\frac{1}{3}} = \frac{2}{3}$	Divide each side by 3.
$\left[(5n-1)^{\frac{1}{3}}\right]^3 = \left(\frac{2}{3}\right)^3$	Cube each side.
$5n - 1 = \frac{8}{27}$	Evaluate the cubes.
$5n = \frac{35}{27}$	Add 1 to each side.
$n = \frac{7}{27}$	Divide each side by 5.

CHECK

$3(5n-1)^{\frac{1}{3}} - 2 = 0$	Original equation
$3\left(5 \cdot \frac{7}{27} - 1\right)^{\frac{1}{3}} - 2 \stackrel{?}{=} 0$	Replace n with $\frac{7}{27}$.
$3\left(\frac{8}{27}\right)^{\frac{1}{3}} - 2 \stackrel{?}{=} 0$	Simplify.
$3\left(\frac{2}{3}\right) - 2 \stackrel{?}{=} 0$	The cube root of $\frac{8}{27}$ is $\frac{2}{3}$.
$0 = 0$ ✓	Subtract.

CHECK Your Progress **Solve each equation.**

2A. $(3n+2)^{\frac{1}{3}} + 1 = 0$ **2B.** $(2y+6)^{\frac{1}{4}} - 2 = 0$

Solve Radical Inequalities A **radical inequality** is an inequality that has a variable in a radicand.

Study Tip

Radical Inequalities

Since a principal square root is never negative, inequalities that simplify to the form $\sqrt{ax+b} \le c$, where c is a negative number, have no solutions.

EXAMPLE 3 Solve a Radical Inequality

Solve $2 + \sqrt{4x - 4} \le 6$.

Since the radicand of a square root must be greater than or equal to zero, first solve $4x - 4 \ge 0$ to identify the values of x for which the left side of the given inequality is defined.

$$4x - 4 \ge 0$$
$$4x \ge 4$$
$$x \ge 1$$

Now solve $2 + \sqrt{4x - 4} \le 6$.

$2 + \sqrt{4x - 4} \le 6$ Original inequality

$\sqrt{4x - 4} \le 4$ Isolate the radical.

$4x - 4 \le 16$ Eliminate the radical.

$4x \le 20$ Add 4 to each side.

$x \le 5$ Divide each side by 4.

It appears that $1 \le x \le 5$. You can test some x-values to confirm the solution. Let $f(x) = 2 + \sqrt{4x - 4}$. Use three test values: one less than 1, one between 1 and 5, and one greater than 5. Organize the test values in a table.

$x = 0$	$x = 2$	$x = 7$
$f(0) = 2 + \sqrt{4(0) - 4}$ $= 2 + \sqrt{-4}$ Since $\sqrt{-4}$ is not a real number, the inequality is not satisfied.	$f(2) = 2 + \sqrt{4(2) - 4}$ $= 4$ Since $4 \le 6$, the inequality is satisfied.	$f(7) = 2 + \sqrt{4(7) - 4}$ ≈ 6.90 Since $6.90 \not\le 6$, the inequality is not satisfied.

The solution checks. Only values in the interval $1 \le x \le 5$ satisfy the inequality. You can summarize the solution with a number line.

CHECK Your Progress

Solve each inequality.

3A. $\sqrt{2x + 2} + 1 \ge 5$ **3B.** $\sqrt{4x - 4} - 2 < 4$

Online Personal Tutor at algebra2.com

CONCEPT SUMMARY *Solving Radical Inequalities*

To solve radical inequalities, complete the following steps.

Step 1 If the index of the root is even, identify the values of the variable for which the radicand is nonnegative.

Step 2 Solve the inequality algebraically.

Step 3 Test values to check your solution.

CHECK Your Understanding

Example 1 (pp. 422–423)

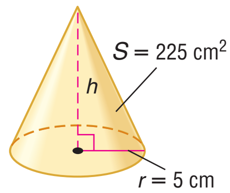
Solve each equation.

1. $\sqrt{4x + 1} = 3$

2. $4 - (7 - y)^{\frac{1}{2}} = 0$

3. $1 + \sqrt{x + 2} = 0$

4. **GEOMETRY** The surface area S of a cone can be found by using $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone. Find the height of the cone.



Example 2 (p. 423)

Solve each equation.

5. $\frac{1}{6}(12a)^{\frac{1}{3}} = 1$

6. $\sqrt[3]{x - 4} = 3$

7. $(3y)^{\frac{1}{3}} + 2 = 5$

Example 3 (p. 424)

Solve each inequality.

8. $\sqrt{2x + 3} - 4 \leq 5$

9. $\sqrt{b + 12} - \sqrt{b} > 2$

10. $\sqrt{y - 7} + 5 \geq 10$

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–22	1
23–30	2
31–38	3

Solve each equation.

11. $\sqrt{x} = 4$

12. $\sqrt{y} - 7 = 0$

13. $a^{\frac{1}{2}} + 9 = 0$

14. $2 + 4z^{\frac{1}{2}} = 0$

15. $7 + \sqrt{4x + 8} = 9$

16. $5 + \sqrt{4y - 5} = 12$

17. $\sqrt{x - 5} = \sqrt{2x - 4}$

18. $\sqrt{2t - 7} = \sqrt{t + 2}$

19. $\sqrt{x - 6} - \sqrt{x} = 3$

20. $\sqrt{y + 21} - 1 = \sqrt{y + 12}$

21. $\sqrt{b + 1} = \sqrt{b + 6} - 1$

22. $\sqrt{4z + 1} = 3 + \sqrt{4z - 2}$

23. $\sqrt[3]{c - 1} = 2$

24. $\sqrt[3]{5m + 2} = 3$

25. $(6n - 5)^{\frac{1}{3}} + 3 = -2$

26. $(5x + 7)^{\frac{1}{5}} + 3 = 5$

27. $(3x - 2)^{\frac{1}{5}} + 6 = 5$

28. $(7x - 1)^{\frac{1}{3}} + 4 = 2$

29. The formula $s = 2\pi\sqrt{\frac{\ell}{32}}$ represents the swing of a pendulum, where s is the time in seconds to swing back and forth, and ℓ is the length of the pendulum in feet. Find the length of a pendulum that makes one swing in 1.5 seconds.

Real-World Link

A ponderal index p is a measure of a person's body based on height h in meters and mass m in kilograms. One such formula is $p = \frac{\sqrt[3]{m}}{h}$.

Source: *A Dictionary of Food and Nutrition*

30. **HEALTH** **Refer to the information at the left.** A 70-kilogram person who is 1.8 meters tall has a ponderal index of about 2.29. How much weight could such a person gain and still have an index of at most 2.5?

Solve each inequality.

31. $1 + \sqrt{7x - 3} > 3$

32. $\sqrt{3x + 6} + 2 \leq 5$

33. $-2 + \sqrt{9 - 5x} \geq 6$

34. $6 - \sqrt{2y + 1} < 3$

35. $\sqrt{2} - \sqrt{x + 6} \leq -\sqrt{x}$

36. $\sqrt{a + 9} - \sqrt{a} > \sqrt{3}$

37. $\sqrt{b - 5} - \sqrt{b + 7} \leq 4$

38. $\sqrt{c + 5} + \sqrt{c + 10} > 2$

39. PHYSICS When an object is dropped from the top of a 50-foot tall building, the object will be h feet above the ground after t seconds, where $\frac{1}{4}\sqrt{50 - h} = t$. How far above the ground will the object be after 1 second?

EXTRA PRACTICE
See pages 907, 932.
Math Online
Self-Check Quiz at algebra2.com

40. FISH The relationship between the length and mass of certain fish can be approximated by the equation $L = 0.46\sqrt[3]{M}$, where L is the length in meters and M is the mass in kilograms. Solve this equation for M.

H.O.T. Problems

41. REASONING Determine whether the equation $\frac{\sqrt{(x^2)^2}}{-x} = x$ is *sometimes*, *always*, or *never* true when x is a real number. Explain your reasoning.

42. Which One Doesn't Belong? Which equation does *not* have a solution?

$\sqrt{x - 1} + 3 = 4$	$\sqrt{x + 1} + 3 = 4$	$\sqrt{x - 2} + 7 = 10$	$\sqrt{x + 2} - 7 = -10$

43. OPEN ENDED Write an equation containing two radicals for which 1 is a solution.

44. CHALLENGE Explain how you know that $\sqrt{x + 2} + \sqrt{2x - 3} = -1$ has no real solution without actually solving it.

45. *Writing in Math* Refer to the information on page 422 to describe how the cost and the number of units manufactured are related. Rewrite the manufacturing equation $C = 10n^{\frac{2}{3}} + 1500$ as a radical equation, and write a step-by-step explanation of how to determine the maximum number of chips the company could make for $10,000.

STANDARDIZED TEST PRACTICE

46. ACT/SAT If $\sqrt{x + 5} + 1 = 4$, what is the value of x?

A 4 **B** 10 **C** 11 **D** 20

47. REVIEW Which set of points describes a function?

F {(3, 0), (−2, 5), (2, −1), (2, 9)}

G {(−3, 5), (−2, 3), (−1, 5), (0, 7)}

H {(2, 5), (2, 4), (2, 3), (2, 2)}

J {(3, 1), (−3, 2), (3, 3), (−3, 4)}

48. REVIEW What is an equivalent form of $\frac{4}{5 + i}$?

A $\frac{10 - 2i}{13}$

B $\frac{5 - i}{6}$

C $\frac{6 - i}{6}$

D $\frac{6 - i}{13}$

Spiral Review

Write each radical using rational exponents. (Lesson 7-6)

49. $\sqrt[7]{5^3}$ **50.** $\sqrt{x+7}$ **51.** $\left(\sqrt[3]{x^2+1}\right)^2$

Simplify. (Lesson 7-5)

52. $\sqrt{72x^6y^3}$ **53.** $\frac{1}{\sqrt[3]{10}}$ **54.** $(5-\sqrt{3})^2$

55. SALES Sales associates at Electronics Unlimited earn \$8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their income, and find how much merchandise they must sell in order to earn \$500 in a 40-hour week. (Lesson 7-2)

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 7-1)

56. $f(x) = x + 5$
$g(x) = x - 3$

57. $f(x) = 10x - 20$
$g(x) = x - 2$

58. $f(x) = 4x^2 - 9$
$g(x) = \frac{1}{2x+3}$

59. ENTERTAINMENT A magician asked a member of his audience to choose any number. He said, "Multiply your number by 3. Add the sum of your number and 8 to that result. Now divide by the sum of your number and 2." The magician announced the final answer without asking the original number. What was the final answer? How did he know what it was? (Lesson 6-4)

Simplify. (Lesson 6-2)

60. $(x+2)(2x-8)$ **61.** $(3p+5)(2p-4)$ **62.** $(a^2+a+1)(a-1)$

CONSTRUCTION For Exercises 63 and 64, use the graph at right that shows the amount of money awarded for construction in Texas. (Lesson 2-5)

63. Let the independent variable be years since 1999. Write a prediction equation from the data for 1999, 2000, 2001, and 2002.

64. Use your prediction equation to predict the amount for 2010.

Money Awarded for Construction in Texas

Source: *Texas Almanac*

Cross-Curricular Project

Algebra and Social Studies

Population Explosion It is time to complete your project. Use the information and data you have gathered about the population to prepare a Web page. Be sure to include graphs, tables, and equations in the presentation.

Math Online Cross-Curricular Project at algebra2.com